Observation of a strange nonchaotic attractor in a neon glow discharge

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A strange nonchaotic attractor is observed in a neon gas discharge without any periodic or quasiperiodic driving force. The estimate of correlation dimension and Lyapunov exponent spectra confirms the existence of the strange nonchaotic attractor. Furthermore, the scaling law of the Fourier amplitude spectrum is observed in agreement with predicted scaling behavior for a quasiperiodically driven strange nonchaotic attractor. [S1063-651X(97)03903-2]

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I. INTRODUCTION

A strange nonchaotic attractor is considered to be a typical property of quasiperiodically forced systems as suggested by Grebogi *et al.* [1] where the term *strange nonchaotic attractor* refers to the attractor that is geometrically complicated, but for which nearby trajectories do not diverge exponentially. Subsequent papers show that strange nonchaotic attractors commonly occur in quasiperiodically driven systems [2–5]. Some experimental observations for such strange nonchaotic attractors are reported [6] in a two-frequency quasiperiodically driven, buckled, magnetoelastic ribbon experiment and in an analog simulator [7]. A strange nonchaotic attractor is not only observed in a quasiperiodically driven system but also in a parametrically excited Duffing equation [8]. On the other hand, quasiperiodic transition to chaos in gas discharge plasma has been investigated experimentally [9–11]. The difficulty [12,13] in the distinction between quasiperiodic attractor and chaotic attractor has been partly overcome by the careful estimate of K_2 entropy, observation of the envelope of the autocorrelation function [9], and the calculation of the Lyapunov exponent spectrum [10]. This offers a possibility to search for strange nonchaotic attractor in plasmas. The identification of a strange nonchaotic attractor depends on its definition, i.e., the nonpositive Lyapunov exponent and fractal dimensionality that reflects the structure of the strange nonchaotic attractor, which is single valued and discontinuous everywhere.

Plasma is a typical nonlinear dynamic system with a large number of degrees of freedom; it is of interest as a medium for testing the universal characteristics of chaos. In the past decade a number of nonlinear dynamic phenomena have been studied in gas discharge plasmas [9,10,14–19]. Quasiperiodic transition to chaos has been frequently observed in plasmas [9–11]. The interplay of these intrinsically different modes contributes to the onset of chaos. The understanding of quasiperiodic attractors and the related strange nonchaotic attractors is significant not only in nonlinear dynamics but also in plasmas. In this paper we present experimental observations on strange nonchaotic attractors that occur in an undriven gas discharge tube, where quasiperiodic oscillations emerge from the plasma itself.

II. EXPERIMENTS

Experiments were performed in a conventional glass neon discharge tube at a pressure of 1.43 torr, radius of 1 cm, and length of 60 cm. The discharge is operated at $i_0 = 2-20$ mA without any external modulation. Self-excited ionization wave patterns are measured by fluctuation of both integrated light flux $F(t)$ and discharge current $i(t)$. These fluctuation signals are digitized by analog to digital convertor (two channels, 12 bits, data length $N = 64\times10^3$). Details of experimental setup (Fig. 1) are described in [20].

It is well known that there are various moving striations in dc discharge tubes over a wide range of discharge parameters [21]. In our experimental conditions there exist p and r waves, which differ in their product of wavelength and electric field strength, and in their spatial modes. The metastable atoms are responsible for the p wave and the atomic ions for the r wave. As already reported [20] the transition from one type of ionization wave to the other is associated with the hysteresis of frequency due to the variation of discharge current. Near the transition points a quasiperiodic oscillation and a strange nonchaotic attractor by either light flux or discharge current can be observed. In a *pure* neon discharge tube without external current modulation we have never observed a chaotic attractor so far. Figure 2 shows light flux $F(t)$ oscillations and their corresponding power spectra at different currents. At $i_0 = 16.07$ mA only a single frequency f_1 can be seen in the spectrum [Fig. 2(d)], which corresponds to a p wave. The trajectory in the phase space was reconstructed by the light flux $(F(t), F(t+\tau), F(t+2\tau))$. The lag time τ represents the fall time (e^{-1}) of the autocorrelation function. The trajectory of motion is a limit cycle [Fig. 2(g)]. With slightly increasing current an incommensurable frequency f_2 (r wave) and some combinations of their har-

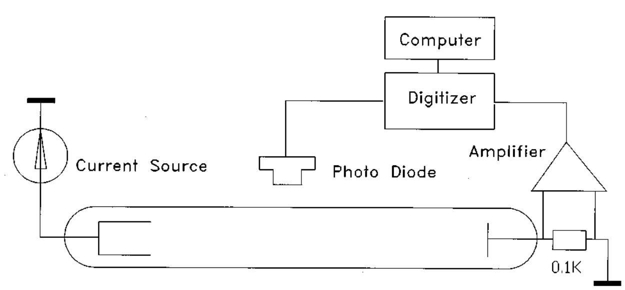

FIG. 1. Geometry of the discharge tube and diagram of the experimental measurements. Integrated light flux $F(t)$ and discharge current $i(t)$ are recorded.

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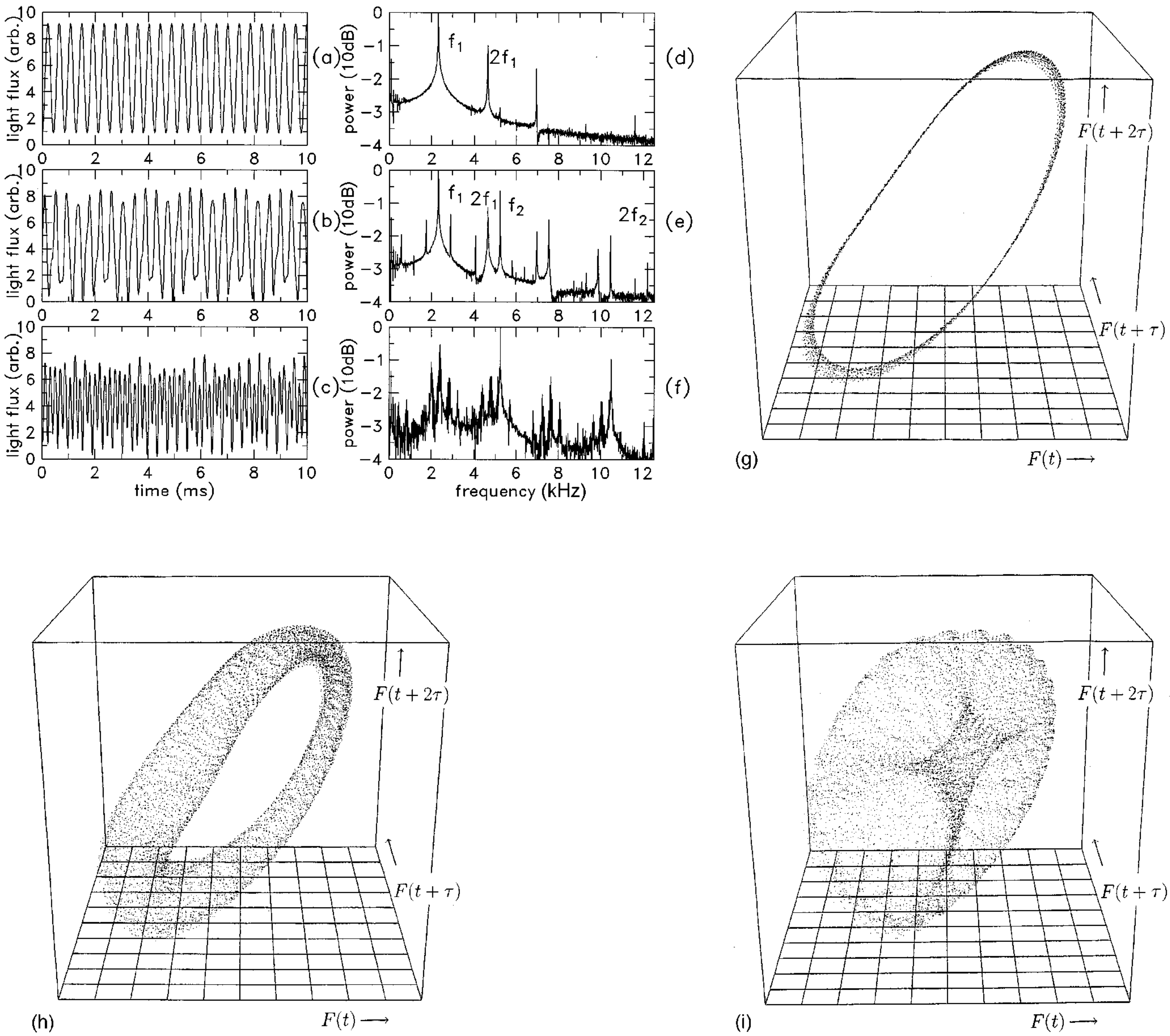

FIG. 2. At $i_0 = 16.07, 16.09, 16.37$ mA, time series, power spectra, and the trajectory in the phase space of the measured integrated light flux $F(t)$ for different attractors. These attractors are identified as periodic (a)(d)(g), quasiperiodic (b)(e)(h), and strange nonchaotic attractors (c)(f)(i).

monic frequencies are seen [Fig. 2(e)] so that a quasiperiodic attractor could be formed. The motion of the trajectory in the phase space is a closed torus, a characteristic of quasiperiodic attractors. With further increasing current abundant harmonics can be found [Fig. 2(i)]. The motion of the trajectory seems to be random in the phase space, the attractor is later found to be a strange nonchaotic attractor rather than a chaotic one. For larger currents, where r wave dominates, periodic oscillation is only observed for f_2. In the transition from p wave to r wave the competition between two waves takes an important role in the formation of a quasiperiodic attractor and a strange nonchaotic attractor.

In order to distinguish those different attractors, let us estimate the dimensionality and Lyapunov exponent spectrum using a conventional phase space reconstruction technique from the time series [10,23]. The Lyapunov exponent is a quantitative measure of the sensitivity of the dynamic system to initial perturbation. Sign and value of the Lyapunov exponents provide a classification of the attractors. Let the Lyapunov exponents λ_i be ordered by size, $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots$. We are able to distinguish the following cases: (1) periodic attractors: $\lambda_1 = 0 > \lambda_2$; (2) two-frequency quasiperiodic attractors: $\lambda_1 = \lambda_2 = 0 > \lambda_3$; (3) three quasiperiodic attractors: $\lambda_1 = \lambda_2 = \lambda_3 = 0 > \lambda_4$; (4) chaotic attractors, $\lambda_1 > 0$; and (5) strange nonchaotic attractors: same as two-frequency quasiperiodic attractors. The strange nonchaotic attractors have the same Lyapunov exponent spectrum as two-frequency quasiperiodic attractors. In order to confirm the strangeness of attractors, we also calculate the correlation dimension. To calculate Lyapunov exponents and correlation dimension we reconstruct a d_m dimension orbit [22]

$$\mathbf{x}_i = (F_i, F_{i+m}, \ldots, F_{i+(d-1)m})$$

TABLE I. Lyapunov exponent spectrum and dimensionality for different discharge currents.

i_0 (mA)	Lyapunov exponents (1/ms)	Dimensionality
16.07	$(-0.001,-0.02,-0.1)$	1.00 ± 0.01
16.09	$(-0.011,-0.005,-0.248,-4.372)$	2.03 ± 0.05
16.37	$(0.19,-0.20,-1.67,-9.20)$	2.23 ± 0.04

from the measured integrated light flux $F(i\Delta t)(i=1,\ldots,N)$, with Δt being the sampling time interval, d_m the embedding dimension, and $\tau=m\Delta t$ the time delay. By choosing $\mathbf{x}_j$ such that $|\mathbf{x}_j-\mathbf{x}_i|\leqslant r$ for small r, the evolution of small vectors $(\mathbf{x}_j-\mathbf{x}_i)$ can be obtained,

$$\mathbf{x}_{j+1}-\mathbf{x}_{i+1}=T_i(\mathbf{x}_j-\mathbf{x}_i).$$

We can decompose the matrix $T_i=Q_iR_i$ into orthogonal matrices Q_i and upper triangular matrices R_i. The Lyapunov exponent λ_l is given by

$$\lambda_l=\frac{1}{K}\sum_{j=0}^{K-1}\ln(R_j)_{ll},\quad l=1,2,\ldots,d_m$$

where K is a given number of matrices T_i. With the series of vectors $\mathbf{x}_i$, the correlation sum $C(\epsilon)$ can be evaluated, defined by

$$C(\epsilon)=\lim_{m\to\infty}\frac{1}{m}\sum_{i,j}^{m}H(\epsilon-|\mathbf{x}_i-\mathbf{x}_j|),$$

where H is the Heaviside function defined by $H(\epsilon)=1$ for positive ϵ, and 0 otherwise. For an intermediate regime of ϵ, $C(\epsilon)$ will scale as $C(\epsilon)\propto\epsilon^d$, therefore the correlation dimension can be obtained by

$$d=\log_{10}C(\epsilon)/\log_{10}\epsilon.$$

These algorithms are applied to the experimental data. Table I summarizes the dimensionality and Lyapunov exponent spectrum for the three sets of data analyzed in this paper. For $i_0=16.07$ mA the attractor is easily identified as $(0,-,-)$, which indicates a periodic attractor. For $i_0=16.09$ mA the exponents are identified as $(0,0,-,-)$ since $|\lambda_1|\approx|\lambda_2|\ll|\lambda_3|,|\lambda_4|$, indicating a two-frequency quasiperiodic attractor. The estimate of dimensionality also indicates that two attractors are periodic and quasiperiodic, respectively. As far as the attractor at $i_0=16.37$ mA is concerned, the trajectory of motion in the phase space [Fig. 2(i)] implies a strange attractor. Its correlation dimension is carefully calculated (Fig. 3) providing a fractal dimension of 2.23. Figure 3 shows log-log plots of $C(\epsilon)$ for the time series at $i_0=16.37$ mA. The slopes of $C(\epsilon)$ on the log-log plot will saturate when the d_m exceeds 6. A little error bar, within ±0.04, is obtained for several experimental runs at $i_0=16.37$ mA. This fractal dimension supports the strangeness of the attractor. However, the attractor is nonchaotic. From the Lyapunov exponent spectrum of the strange attractor at $i_0=16.37$ mA one can obtain a positive Lyapunov exponent $\lambda_1=0.19$, which at first glance implies that the attractor may be recognized as a chaotic one. However, if we

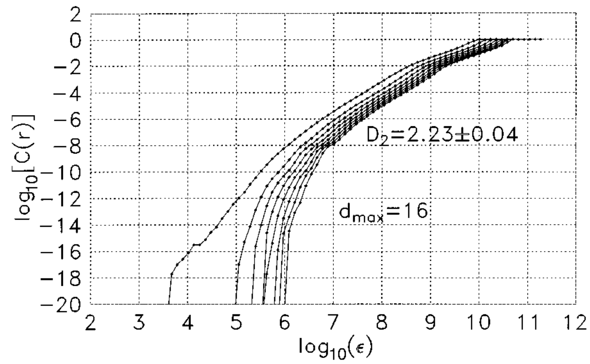

FIG. 3. Correlation dimension $D_2=2.23\pm0.04$ for time series Fig. 2(c). $d_{\max}$ is a maximum of embedding dimension.

recognize λ_2 as zero, since at least one Lyapunov exponent has to be zero for a chaotic attractor, we cannot find much difference in the magnitude of order between λ_1,λ_2, and $\max(|\lambda_1|,|\lambda_2|,)\ll|\lambda_3|$. On the other hand, unavoidable experimental noise causes the computed Lyapunov exponents to become larger [10]. Therefore λ_1,λ_2 could be identified as $(0,0,-)$, showing that the attractor is nonchaotic. The nonchaotic behavior of this attractor is further supported by the observation of an autocorrelation function as shown in Fig. 4, where the autocorrelation function has no significant decay as would be expected for a chaotic attractor (also discussed in [9]). The autocorrelation is similar to that of the usual quasiperiodic motion [24]. The complicated geometric structure in the phase space and the fractal dimension thus indicate the strangeness of the attractor, and Lyapunov exponent spectrum and autocorrelation function indicate a nonchaotic attractor. In general, strange nonchaotic attractors typically appear in quasiperiodically forced nonlinear systems. The strange nonchaotic attractor observed in plasmas is associated with the competition among those intrinsically independent wave modes.

In order to understand the strange nonchaotic attractor in our case, we present its spectral distribution function $N(\sigma)$ as the number of peaks of Fourier spectra with amplitudes greater than σ [3,6]. The distinct scaling relation for $N(\sigma)$ has been investigated [2,4], being $N(\sigma)\propto\ln(\sigma)$ for two-frequency quasiperiodic attractors, $N(\sigma)\propto[\ln(\sigma)]^2$ for three-frequency quasiperiodic attractors, and $N(\sigma)\propto\sigma^{-\alpha}$, $1<\alpha<2$ for strange nonchaotic attractors. These results are plotted in Fig. 5. The approximately straight line in the log-

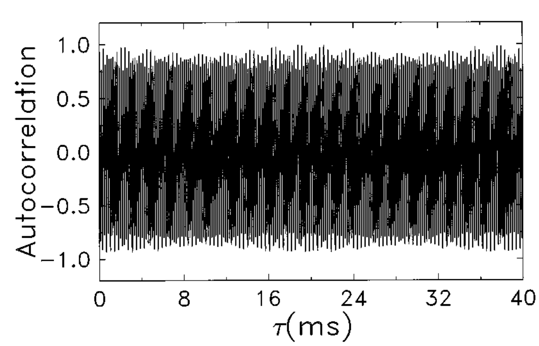

FIG. 4. Autocorrelation function of time series in Fig. 2(c). Strange nonchaotic attractor at $i_0=16.37$ mA.

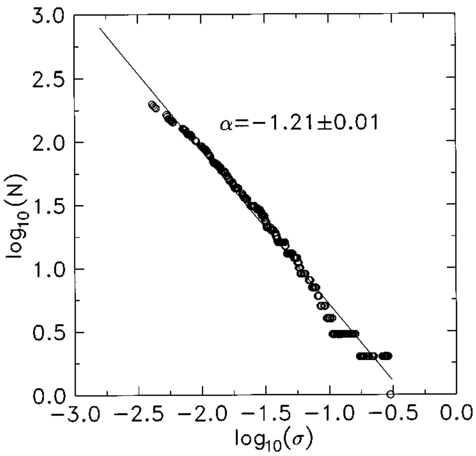

FIG. 5. Spectral distribution $N(\sigma)\propto\sigma^{-\alpha}$ of a strange nonchaotic attractor [Fig. 2(f), $i_0=16.37$ mA].

log plot indicates the power-law relationship, $N(\sigma)\sim\sigma^{-\alpha}$. A least-squares fit yields $\alpha=1.21\pm0.01$. The scaling law agrees with that of the strange nonchaotic attractor predicted for the quasiperiodically driven nonlinear system, further suggesting the existence of a strange nonchaotic attractor. As can be seen from Fig. 2(e) quasiperiodical motion is dominated by p waves and r waves. The harmonic peaks are not abundant so that statistics of spectral distribution is poor where noise dominates.

III. DISCUSSION AND CONCLUSION

Strange nonchaotic attractors are not only observed in quasiperiodically driven systems, but also in an undriven plasma system. Physically there exist two ionization wave patterns in the discharge tube. The coupling of oscillations can be realized by feedback mechanisms through an external circuit. Two independent wave patterns drive a nonlinear plasma medium with a large number of degrees of freedom, which takes an important role in the formation of strange nonchaotic attractor. The two dominant modes are similar to the quasiperiodically driving force as studied in these quasiperiodically driven systems. The frequencies of oscillations are well defined by two different relaxation processes.

A strange nonchaotic attractor in a neon discharge tube has been found where no external quasiperiodically driven force is applied. An analysis of fractal dimension, Lyapunov exponents, autocorrelation function, and Fourier spectra amplitude distributions supports the experimental observation of strange nonchaotic attractor in a neon discharge plasma.

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[1] C. Grebogi, E. Ott, S. Pelikan, and J.A. York, Physica D **13**, 261 (1984).

[2] A. Bondeson, E. Ott, and T.M. Antonsen, Jr., Phys. Rev. Lett. **55**, 2103 (1985).

[3] F.J. Romeiras and E. Ott, Phys. Rev. A **35**, 4404 (1987); F.J. Romeiras, A. Bondeson, and E. Ott, Physica D **26**, 277 (1987).

[4] M. Ding, C. Grebogi, and E. Ott, Phys. Rev. A **39**, 2593 (1989).

[5] Ying-Cheng Lai, Phys. Rev. E **53**, 57 (1996).

[6] W.L. Ditto, M.L. Spano, H.T. Savage, S.N. Rauseo, J. Heagy, and E. Ott, Phys. Rev. Lett. **65**, 533 (1990).

[7] Ting Zhou, F. Moss, and Adi Bulsara, Phys. Rev. A **45**, 5394 (1992).

[8] Tomasz Kapitaniak, *Chaotic Oscillations in Mechanical Systems* (Manchester University Press, Manchester, 1991).

[9] B. Albrecht, H. Deutsch, R.W. Leven, and C. Wilke, Phys. Scr. **47**, 196 (1993); C. Wilke, H. Deutsch, and R.W. Leven, Contrib. Plasma Phys. **30**, 659 (1990).

[10] W.X. Ding, W. Huang, X.D. Wang, and C.X. Yu, Phys. Rev. Lett. **70**, 170 (1993).

[11] S.H. Fan, S.Z. Yang, J.H. Dai, S.B. Zheng, D.P. Yuan, and S.T. Tsai, Phys. Lett. A **164**, 295 (1992).

[12] A.W. Cumming and P.S. Linsay, Phys. Rev. Lett. **60**, 2719 (1988).

[13] P.S. Linsay and A.W. Cumming, Physica D **40**, 1196 (1989).

[14] T. Braun, J.A. Lisboa, R.E. Francke, and J.A.C. Gallas, Phys. Rev. Lett. **59**, 613 (1987).

[15] P.Y. Cheung and A.Y. Wong, Phys. Rev. Lett. **59**, 551 (1987).

[16] J. Qin, L. Wang, D.P. Yuan, P. Gao, and B.Z. Zhang, Phys. Rev. Lett. **63**, 163 (1989); D.L. Feng, J. Zheng, W. Huang, C.X. Yu, and W.X. Ding, Phys. Rev. E **54**, 2839 (1996).

[17] F. Greiner, T. Klinger, H. Klostermann, and A. Piel, Phys. Rev. Lett. **70**, 3071 (1993).

[18] F. Greiner, T. Klinger, and A. Piel, Plasma Phys. **2**, 1810 (1995); T. Klinger, F. Greiner, A. Rohde, and A. Piel, *ibid.* **2**, 1822 (1995).

[19] T. Ueki, Y. Nishida, and N. Yugami, Phys. Rev. Lett. **76**, 4171 (1996).

[20] K.-D. Weltmann, H. Deutsch, H. Unger, and C. Wilke, Contrib. Plasma Phys. **33**, 73 (1993).

[21] N.L. Oleson and A.W. Cooper, Adv. Electron. Phys. **24**, 155 (1968).

[22] N.H. Packard, J.P. Gruchfiled, J.D. Farmar, and R.S. Shaw, Phys. Rev. Lett. **45**, 712 (1980).

[23] P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983); M. Sano and Y. Sawada, *ibid.* **55**, 1082 (1985); A. Wolf, J.B. Swift, H.L. Swinney, and J.A. Vastano, Physica D **16**, 285 (1985); W. Huang, W.X. Ding, D.L. Feng, and C.X. Yu, Phys. Rev. E **50**, 1062 (1994).

[24] Arkady S. Pikovsky and Ulrike Feudel, J. Phys. A **27**, 5209 (1994).